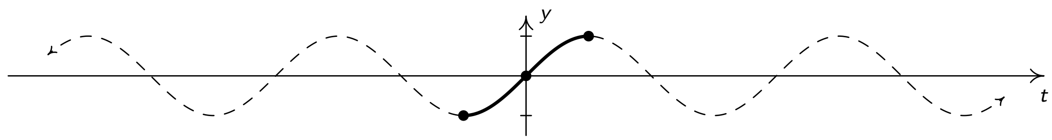
MATH 1700: SECTION 11.3: THE INVERSE CIRCULAR FUNCTIONS

INVERTING THE SINE AND COSINE FUNCTIONS:

In this section we concern ourselves with finding inverses of the circular (trigonometric) functions. Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domains of the circular functions.

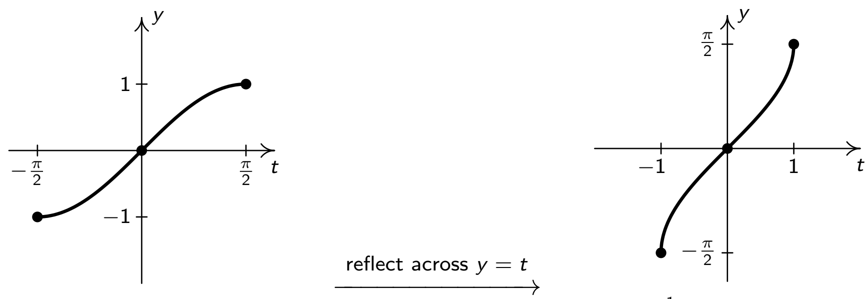
We start with $f(t) = \sin(t)$ and restrict our domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in order to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.



Restricting the domain of $f(t) = \sin(t)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

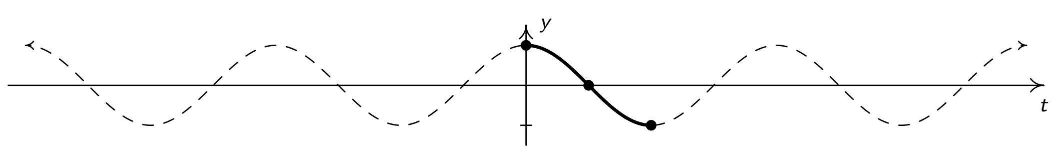
Recall that the inverse of a function f is typically denoted f^{-1}. For this reason, some textbooks use the notation $f^{-1}(t) = \sin^{-1}(t)$ for the inverse of $f(t) = \sin(t)$. Seeing as the convention of writing $(\sin(t))^2$ as $\sin^2(t)$, $(\sin(t))^3$ as $\sin^3(t)$ and so on, it's easy to confuse $\sin^{-1}(t)$ with $\frac{1}{\sin(t)} = \csc(t)$ so we will not use this notation.

Instead, we use the notation $f^{-1}(t) = \arcsin(t)$, read 'arc-sine of t'. We'll explain the 'arc' in 'arcsine' shortly. We graph $f(t) = \sin(t)$ and $f^{-1}(t) = \arcsin(t)$ by reflecting the former across the line $y = t$:



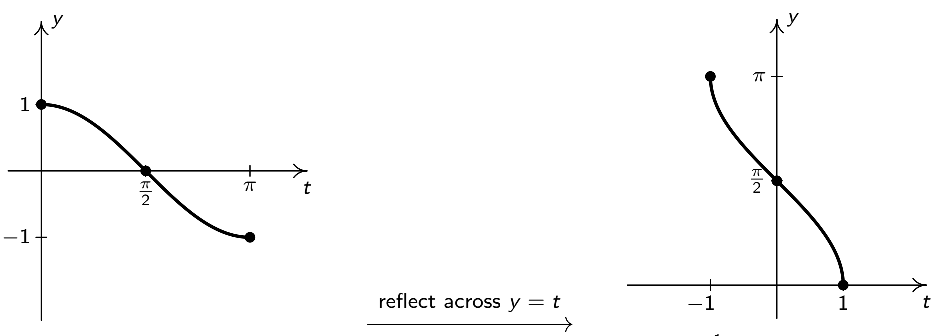
$f(t) = \sin(t)$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. $f^{-1}(t) = \arcsin(t)$.

Next, we consider $g(t) = \cos(t)$. Here, we select the interval $[0, \pi]$ for our restriction.



Restricting the domain of $f(t) = \cos(t)$ to $[0, \pi]$.

Reflecting the across the line $y = t$ produces the graph $y = g^{-1}(t) = \arccos(t)$.



$f(t) = \cos(t)$, $0 \le t \le \pi$ $f^{-1}(t) = \arccos(t)$.

PROPERTIES OF THE ARCSINE AND ARCCOSINE FUNCTIONS:

- Properties of $F(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $\sin(t) = x$ and $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $\sin(\arcsin(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arcsin(\sin(t)) = t$ provided $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $F(x) = \arcsin(x)$ is odd
- Properties of $G(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $\cos(t) = x$ and $0 \leq t \leq \pi$
  - $\cos(\arccos(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arccos(\cos(t)) = t$ provided $0 \leq t \leq \pi$

Before moving to an example, we take a moment to understand the 'arc' in 'arcsine.'

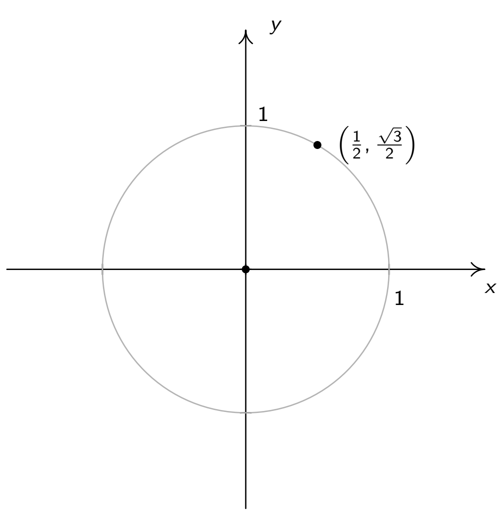
Consider the figure below which illustrates the specific case of $\arcsin\left(\frac{\sqrt{3}}{2}\right)$.

By definition, the real number $t = \arcsin\left(\frac{\sqrt{3}}{2}\right)$ satisfies $\sin(t) = \frac{\sqrt{3}}{2}$ with $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

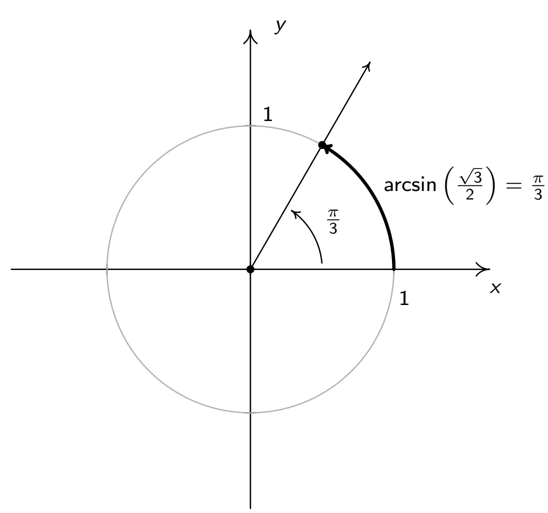
Hence, we are looking for angle measuring t radians between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with a sine of $\frac{\sqrt{3}}{2}$, so $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

In terms of oriented arcs, if we start at $(1, 0)$ and travel along the Unit Circle in the positive (counterclockwise) direction for $\frac{\pi}{3}$ units, we will arrive at the point whose y-coordinate is $\frac{\sqrt{3}}{2}$.

Hence, the real number $\frac{\pi}{3}$ also corresponds to 'arc' corresponding to the 'sine' that is $\frac{\sqrt{3}}{2}$.



y-value on Unit Circle is $\frac{\sqrt{3}}{2}$.



corresponding oriented arc: $t = \frac{\pi}{3}$

In general, the function $f(t) = \sin(t)$ takes a real number input t, associates it with the angle $\theta = t$ radians, and returns the value $\sin(\theta)$. The value $\sin(\theta) = \sin(t)$ is the y-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1, 0)$.

Hence, we may view the inputs to $f(t) = \sin(t)$ as oriented arcs and the outputs as y-coordinates on the Unit Circle. Therefore, the function f^{-1} reverses this process and takes y-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arcsine.

EXAMPLE 1:

1. Find the exact values of the following.

   (a) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

   (b) $\arccos\left(\frac{1}{2}\right)$

   (c) $\arcsin\left(-\frac{1}{2}\right)$

   (d) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

   (e) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$

   (f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$

   (g) $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$

   (h) $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

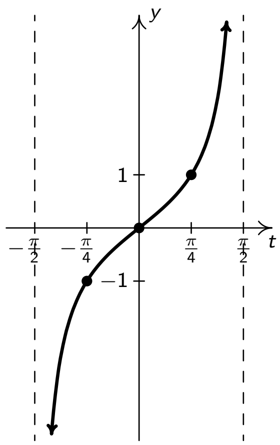
2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $f(x) = \tan(\arccos(x))$

   (b) $g(x) = \cos(2\arcsin(x))$

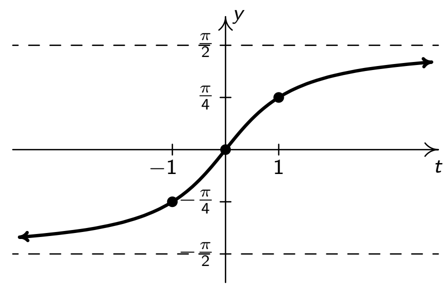
INVERTING THE TANGENT AND COTANGENT FUNCTIONS:

The next pair of functions we wish to discuss are the inverses of tangent and cotangent. First, we restrict $f(t) = \tan(t)$ to its fundamental cycle on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain the arctangent function, $f^{-1}(t) = \arctan(t)$. Among other things, note that the *vertical* asymptotes $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$ of the graph of $f(t) = \tan(t)$ become the *horizontal* asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(t) = \arctan(t)$.



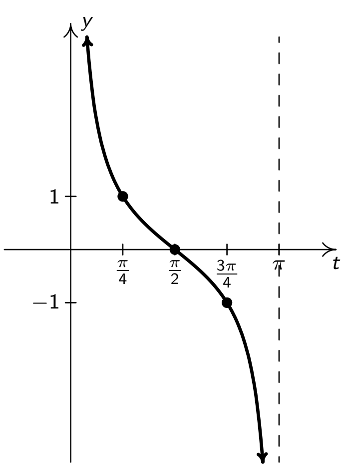
$f(t) = \tan(t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

reflect across $y = t$ $\longrightarrow$ switch t and y coordinates



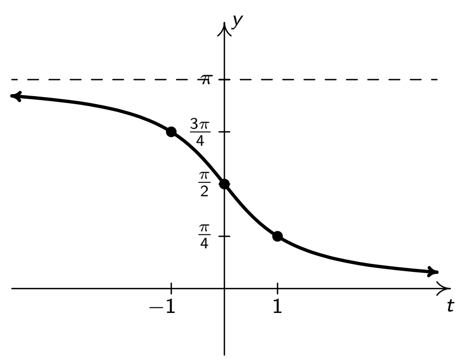
$f^{-1}(t) = \arctan(t)$.

Next, we restrict $g(t) = \cot(t)$ to its fundamental cycle on $(0, \pi)$ to obtain $g^{-1}(t) = \operatorname{arccot}(t)$, the arccotangent function. Once again, the vertical asymptotes $t = 0$ and $t = \pi$ of the graph of $g(t) = \cot(t)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ of the graph of $g^{-1}(t) = \operatorname{arccot}(t)$.



$g(t) = \cot(t)$, $0 < t < \pi$.

reflect across $y = t$ $\longrightarrow$ switch t and y coordinates



$g^{-1}(t) = \operatorname{arccot}(t)$.

PROPERTIES OF THE ARCTANGENT AND ARCCOTANGENT FUNCTIONS:

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}^{+}$; as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}^{-}$
  - $\arctan(x) = t$ if and only if $\tan(t) = x$ and $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $\arctan(x) = \operatorname{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(t)) = t$ provided $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $F(x) = \arctan(x)$ is odd
- Properties of $G(x) = \operatorname{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - as $x \to -\infty$, $\operatorname{arccot}(x) \to \pi^{-}$; as $x \to \infty$, $\operatorname{arccot}(x) \to 0^{+}$
  - $\operatorname{arccot}(x) = t$ if and only if $\cot(t) = x$ and $0 < t < \pi$
  - $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(t)) = t$ provided $0 < t < \pi$

EXAMPLE 2:

1. Find the exact values of the following.

   (a) $\arctan(\sqrt{3})$

   (b) $\operatorname{arccot}(-\sqrt{3})$

   (c) $\cot(\operatorname{arccot}(-5))$

   (d) $\sin\left(\arctan\left(-\frac{4}{3}\right)\right)$

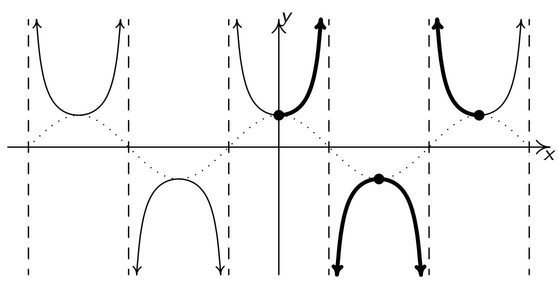
2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.

   (a) $\tan(2\arctan(x))$

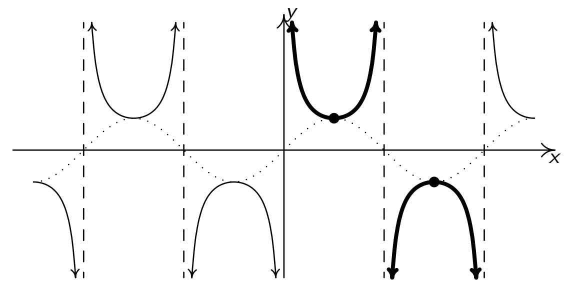
   (b) $\cos(\operatorname{arccot}(2x))$

INVERTING THE SECANT AND COSECANT FUNCTIONS:

It is clear from the graphs of secant and cosecant that we cannot find one single continuous piece of its graph which covers the entire range of $(-\infty, -1] \cup [1, \infty)$ and restricts the domain of the function so that it is one-to-one:



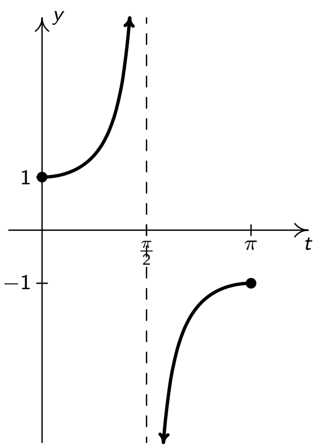
The graph of $y = \sec(x)$.



The graph of $y = \csc(x)$.

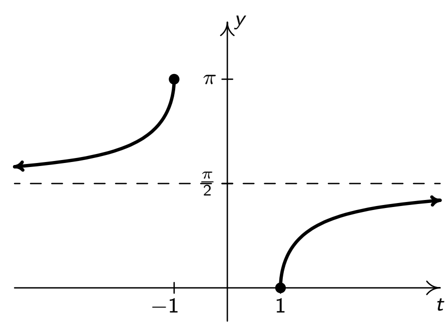
Thus in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$. There are two generally accepted ways make these choices which restrict the domains of these functions so that they are one-to-one. The text has subsections which cover both cases. In this class, we'll emphasize the choice which works for our classes here at Lakeland.

We restrict $f(t) = \sec(t)$ to a domain of $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$:



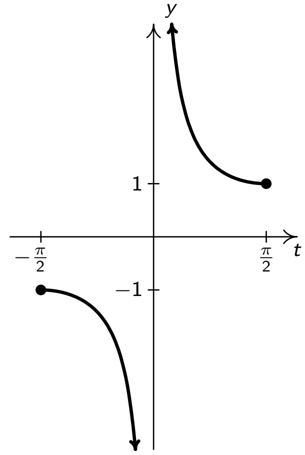
$f(t) = \sec(t)$ on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

$\xrightarrow[\text{switch } t \text{ and } y \text{ coordinates}]{\text{reflect across } y = t}$



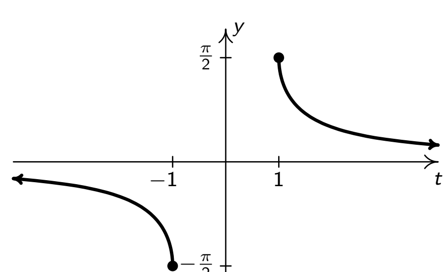
$f^{-1}(t) = \text{arcsec}(t)$

We restrict $g(t) = \csc(t)$ to a doman of $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.



$g(t) = \csc(t)$ on $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$\xrightarrow[\text{switch } t \text{ and } y \text{ coordinates}]{\text{reflect across } y = t}$



$g^{-1}(t) = \text{arccsc}(t)$

PROPERTIES OF THE ARCSECANT AND ARCCOSECANT FUNCTIONS

- Properties of $F(x) = \text{arcsec}(x)$
  - Domain: $\{x \,|\, |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
  - as $x \to -\infty$, $\text{arcsec}(x) \to \frac{\pi}{2}^{+}$; as $x \to \infty$, $\text{arcsec}(x) \to \frac{\pi}{2}^{-}$
  - $\text{arcsec}(x) = t$ if and only if $\sec(t) = x$ and $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
  - $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\sec\left(\text{arcsec}(x)\right) = x$ provided $|x| \geq 1$
  - $\text{arcsec}(\sec(t)) = t$ provided $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$
- Properties of $G(x) = \text{arccsc}(x)$
  - Domain: $\{x \,|\, |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
  - as $x \to -\infty$, $\text{arccsc}(x) \to 0^{-}$; as $x \to \infty$, $\text{arccsc}(x) \to 0^{+}$
  - $\text{arccsc}(x) = t$ if and only if $\csc(t) = x$ and $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\csc\left(\text{arccsc}(x)\right) = x$ provided $|x| \geq 1$
  - $\text{arccsc}(\csc(t)) = t$ provided $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$
  - $G(x) = \text{arccsc}(x)$ is odd

EXAMPLE 3:

1. Find the exact values of the following.
   (a) $\text{arcsec}(2)$
   (b) $\text{arccsc}(-2)$
   (c) $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$
   (d) $\cot\left(\text{arccsc}\left(-3\right)\right)$
2. Rewrite each of the following composite functions as algebraic functions of x and state the domain.
   (a) $f(x) = \tan(\text{arcsec}(x))$
   (b) $g(x) = \cos(\text{arccsc}(4x))$

HOMEWORK: Section 11.3: Page 969: every other odd: 1 - 45, 57 - 117, and 133 - 185.

SOLVING EQUATIONS WITH THE INVERSE CIRCULAR FUNCTIONS:

EXAMPLE 4: Solve the following equations.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.

2. Find all real numbers t for which $\tan(t) = -2$

3. Solve $\sec(x) = -\frac{5}{3}$ for x.

EXAMPLE 5: Consider the function $f(t) = 3\cos(6t) - 4\sin(6t)$. Find a formula for $f(t)$:

1. in the form $C(t) = A\cos(\omega t + \phi) + B$ for $\omega > 0$

2. in the form $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$

HOMEWORK: Section 11.3: Page 973: 189 - 213 odd, 228, 230*

INVERSE CIRCULAR FUNCTION SUMMARY

- $\sin^{-1}(x) = \arcsin(x)$ is an angle between $-\pi/2$ and $\pi/2$ (Quadrant I or $-$IV) whose sine is x.
- $\csc^{-1}(x) = \text{arccsc}(x)$ is an angle between $-\pi/2$ and $\pi/2$ (Quadrant I or $-$IV) whose cosecant is x.
- **NOTE:** $\text{arccsc}(x) = \arcsin(1/x)$, for $x \neq 0$.
- $\tan^{-1}(x) = \arctan(x)$ is an angle between $-\pi/2$ and $\pi/2$ (Quadrant I or $-$IV) whose tangent is x.
- **NOTE:** $\arcsin(x)$, $\text{arccsc}(x)$, and $\arctan(x)$ are odd functions:

$$\arcsin(-x) = -\arcsin(x) \quad \text{arccsc}(-x) = -\text{arccsc}(x) \quad \arctan(-x) = -\arctan(x)$$

- $\cos^{-1}(x) = \arccos(x)$ is an angle between 0 and π (Quadrant I or II) whose cosine is x.
- $\sec^{-1}(x) = \text{arcsec}(x)$ is an angle between 0 and π (Quadrant I or II) whose secant is x.
- **NOTE:** $\text{arcsec}(x) = \arccos(1/x)$, for $x \neq 0$.
- $\cot^{-1}(x) = \text{arccot}(x)$ is an angle between 0 and π (Quadrant I or II) whose cotangent is x.
- **NOTE:** $\text{arccot}(x) = \arctan(1/x)$ **ONLY WHEN** $x > 0$